



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 46 Random Walks

J P Singh

Department of Management Studies



ONE STEP UNBIASED RANDOM WALK

- Consider a **one step** stochastic process $W_1(T)$ whose initial state ($t=0$) is represented by the origin ($X=0$ at $t=0$).
- Since this is a one step process involving only one jump at $t=T$, it can be modelled by one random variable X_1 .



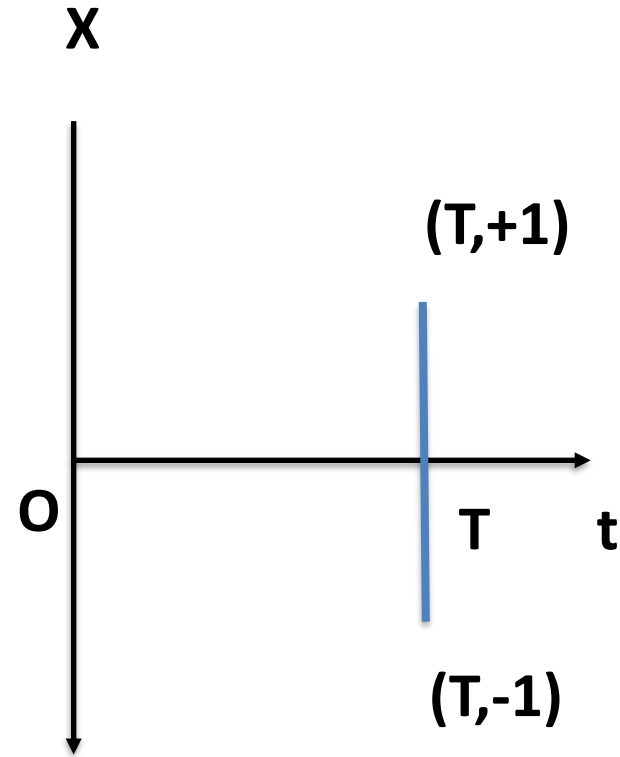
- Let time be modelled discretely with time step T so that the first time step is at $t=T$;
- At this point ($t=T$) the variable X_1 will make its first and only move.
- This move is random up or down with probability $\frac{1}{2}$ of magnitude 1 unit.
- Hence, the variable X_1 is discrete & can take one of two values ± 1 with probability $\frac{1}{2}$ each i.e. it is UNBIASED.

Then,

$$W_1(T) = X_1 = \pm 1;$$

$$E[W_1(T)] = E(X_1) = 0,$$

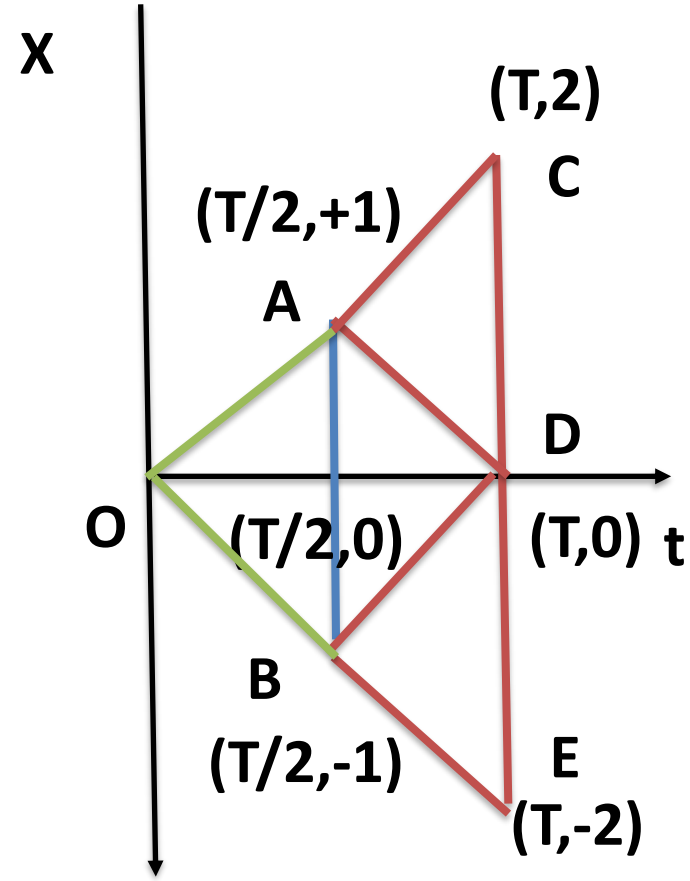
$$E(X_1^2) = 1, \sigma_1^2 = 1.$$



2 STEP RANDOM WALK

- Let us, now, assume that the time step size be reduced to $T/2$ so that there are two time steps in $(0,T)$.
- Let X_1 represent the random jump at the first time step and let X_2 represent the random jump at the second time step.
- $W_2(T)=W_2(T/2)+X_2=X_1+X_2$;
 $E[W_2(T)]=E(X_1+X_2)=0$,
- $E[W_2(T)]^2=E(X_1+X_2)^2=E(X_1)^2+E(X_2)^2+2E(X_1X_2)=2$
- $E(X_1X_2)=E(X_1)E(X_2)=0$
- $\sigma_T^2=2$
since X_1 & X_2 are uncorrelated.

$$\begin{aligned}
 W_2(T) &= X_1 + X_2 = 0, \pm 2; \\
 E[W_2(T)] &= E(X_1 + X_2) = 0 \\
 E[W_2(T)]^2 &= E(X_1 + X_2)^2 = \\
 E(X_1)^2 + E(X_2)^2 + 2E(X_1 X_2) &= 2, \\
 \sigma_T^2 &= 2.
 \end{aligned}$$



GENERALIZED UNBIASED UNSCALED RANDOM WALK

$$\begin{aligned}\mathbf{W}_n(\mathbf{T}) &= \mathbf{W}_n\left(\mathbf{n} - 1, \frac{\mathbf{T}}{\mathbf{n}}\right) + \mathbf{X}_n \\ &= \mathbf{W}_n\left(\mathbf{n} - 2, \frac{\mathbf{T}}{\mathbf{n}}\right) + \mathbf{X}_{n-1} + \mathbf{X}_n = \sum_{i=1}^n \mathbf{X}_i \\ \mathbf{E}[\mathbf{W}_n(\mathbf{T})] &= \mathbf{E}\left[\sum_{i=1}^n \mathbf{X}_i\right] = \left[\sum_{i=1}^n \mathbf{E}(\mathbf{X}_i)\right] = \mathbf{0} \\ \mathbf{E}[\mathbf{W}_n(\mathbf{T})]^2 &= \mathbf{E}\left[\sum_{i=1}^n \mathbf{X}_i\right]^2 = \mathbf{E}\left[\sum_{i=1}^n \mathbf{X}_i^2\right] = \left[\sum_{i=1}^n \mathbf{E}(\mathbf{X}_i^2)\right] = \mathbf{n} \\ \sigma_{\mathbf{T}}^2 &= \mathbf{n}\end{aligned}$$

- What is n . n is the number of steps. Hence, what happens when n increases. The step size decreases. And we move towards the continuous regime.
- It seems that as we move to the continuous time case, the variance of the process diverges. The process tends to blow up.
- This makes the unscaled random walk of very limited utility in modeling physical and financial stochastic systems.
- So what do we do? We see that the variance is growing as n , the number of steps.
- Hence, if we rescale X 's to $X \rightarrow X/\sqrt{n}$, we are done.

- Are we? The variance, now, becomes a round one for all n and T i.e. it is independent of the process length.
- It is, however, the case that the variance of stochastic processes increases with the time of evolution. We need to factor in this issue into our definition.
- Hence, we use the scaling $X \rightarrow Y = X\sqrt{T/n}$.

Hence, our complete set of assumptions is :

(i) $W_n(0) = 0$

(ii) no of steps = n so that, layer spacing T/n ,

(iii) up and down jumps equal and of size $\sqrt{\frac{T}{n}}$,

(iv) up and down probabilities everywhere equal to $\frac{1}{2}$.



Mathematically, our random walk consists of n Independent identically distributed random variables Y_i

$$Y_i = \sqrt{\frac{T}{n}} X_i \text{ where } X_i \text{ are IIDs defined by}$$

$$X_i = \begin{cases} +1 & \text{with } p(X_i = +1) = 1/2 \\ -1 & \text{with } p(X_i = -1) = 1/2 \end{cases}$$

and the recursive relation

$$\begin{aligned} W_n^Y(T) &= W_n^Y\left(n \cdot \frac{T}{n}\right) = W_n^Y\left((n-1) \cdot \frac{T}{n}\right) + Y_n \\ &= W_n^Y\left((n-2) \cdot \frac{T}{n}\right) + Y_{n-1} + Y_n = \sum_{i=1}^n Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^n X_i \end{aligned}$$

$$\mathbf{E}\left[\mathbf{W}_n^Y(\mathbf{T})\right] = \mathbf{E}\left[\sum_{i=1}^n \mathbf{Y}_i\right] = \left[\sum_{i=1}^n \mathbf{E}(\mathbf{Y}_i)\right] = \mathbf{0}$$

$$\mathbf{E}\left[\mathbf{W}_n^Y(\mathbf{T})\right]^2 = \mathbf{E}\left[\sum_{i=1}^n \mathbf{Y}_i\right]^2 = \mathbf{E}\left[\sum_{i=1}^n \mathbf{Y}_i^2\right] = \left[\sum_{i=1}^n \mathbf{E}(\mathbf{Y}_i^2)\right]$$

$$\left[\sum_{i=1}^n \mathbf{E}\left(\frac{\mathbf{T}}{\mathbf{n}} \mathbf{X}_i^2\right)\right] = \frac{\mathbf{T}}{\mathbf{n}} \left[\sum_{i=1}^n \mathbf{E}(\mathbf{X}_i^2)\right] = \frac{\mathbf{T}}{\mathbf{n}} \mathbf{n} = \mathbf{T}. \text{ Hence } \sigma_{\mathbf{T}}^2 = \mathbf{T}$$

Clearly, $E[W_n^Y(T)] = 0$; $\text{Var}[W_n^Y(T)] = T$

Please note so far no limits have been taken.

For an arbitrary t in $(0, T)$, no of steps $= \frac{nt}{T}$. Hence,

In analogy with $W_n^Y(T) = W_n^Y\left(n \cdot \frac{T}{n}\right) = \sum_{i=1}^n Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^n X_i$

we have $W_n^Y(t) = W_n^Y\left(\frac{nt}{T} \cdot \frac{T}{n}\right) = \sum_{i=1}^{nt/T} Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^{nt/T} X_i$

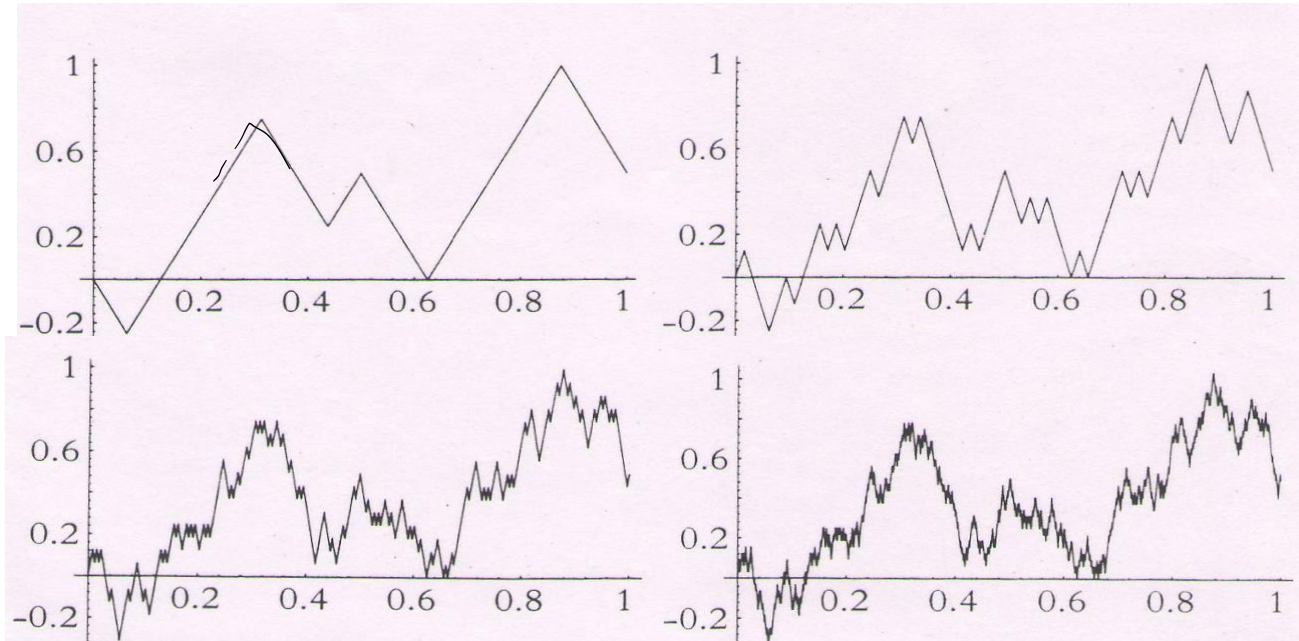
$E[W_n^Y(t)] = 0$; $E[W_n^Y(t)]^2 = \frac{T}{n} \cdot \frac{nt}{T} = t$; $\text{Var}[W_n^Y(t)] = t$

WHAT IS $W_n(t)$

- $W_n(T/n)$ is the value of the process at the first layer i.e. $(T/n)^{(1/2)}$ or $-(T/n)^{(1/2)}$.
- Similarly, $W_n(t) = W_n[(nt/T)T/n]$ is the spectrum of possible values of the process at the nt/T^{th} layer i.e. at the end of time t .



RANDOM WALK OF 16,64,256 AND 1024 STEPS RESPECTIVELY



- To move from the above discrete version of random walk to the continuous time version (formally called Brownian motion) several approaches can be adopted e.g.
- The Central Limit Theorem;
- The Diffusion PDE;
- The Stirling Approximation.